Series RLC circuit

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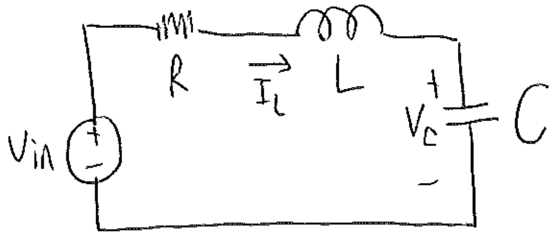


differential equations

$$\frac{d^2V_c}{dt^2}+\frac{R}{L}\frac{dV_c}{dt}+\frac{1}{LC}V_c=\frac{1}{LC}V_{in}(t)$$

$$\frac{d^2I_L}{dt^2}+\frac{R}{L}\frac{dI_L}{dt}+\frac{1}{LC}I_L=\frac{1}{L}\frac{dV_{in}(t)}{dt}$$

characteristic equation

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0 \qquad \text{let} \quad 2\alpha=\frac{R}{L}\geq 0 \quad w_0^2=\frac{1}{LC}>0$$

$\alpha \to$ damping ratio
$w_0 \to$ resonant frequency

hence characteristic equation $s^2+2\alpha s+w_0^2=0$

roots of characteristic equation $s_{1,2}=-\alpha \mp \sqrt{\alpha^2-w_0^2}$ (natural frequencies)

1 Assume $V_{in}(t)=1$ Volt

$$\frac{d^2V_c}{dt^2}+2\alpha\frac{dV_c}{dt}+w_0^2V_c=w_0^2\cdot 1 \longrightarrow V_{cp}=1 \qquad V_c(t)=V_{cp}+V_{ch}$$

$$\frac{d^2I_L}{dt^2}+2\alpha\frac{dI_L}{dt}+w_0^2I_L=\frac{1}{L}\cdot 0 \longrightarrow I_{Lp}=0 \qquad I_L(t)=I_{Lp}+I_{Lh}$$

* Overdamped case $[\alpha > w_0]$ $\quad \frac{R}{2L}>\frac{1}{\sqrt{LC}}$ $\quad w_d \triangleq \sqrt{\alpha^2-w_0^2}$ $\quad s_{1,2}=-\alpha\mp\sqrt{\alpha^2-w_0^2}$

$\alpha > \sqrt{\alpha^2-w_0^2} > 0 \longrightarrow$ hence $\to s_1=-\alpha+w_d<0$, $\to s_2=-\alpha-w_d<0$

(both natural frequencies are real and negative)

$$V_c(t)=1+A_1e^{s_1t}+A_2e^{s_2t} \qquad \lim_{t\to\infty}V_c(t)=1$$

$$I_L(t)=B_1e^{s_1t}+B_2e^{s_2t} \qquad \lim$$

* Critically damped case $[\alpha=w_0]$ $\quad \frac{R}{2L}=\frac{1}{\sqrt{LC}}$ $\quad s_{1,2}=-\alpha$ (real, negative double root)

$-\alpha<0$

$$V_c(t)=1+(A_1+A_2t)e^{-\alpha t}$$

$$I_L(t)=(B_1+B_2t)e^{-\alpha t}$$

(2)

* Underdamped Case $\alpha < w_0$ $w_d = \sqrt{w_0^2 - \alpha^2}$ → frequency (angular frequency) oscillations of damped natural frequencies

$s_{1,2} = -\alpha \mp j w_d$ [complex roots] $Re\{s_{1,2}\} < 0$

$V_c(t) = 1 + e^{-\alpha t}\left[A_1 \cos(w_d t) + A_2 \sin(w_d t)\right]$

$I_L(t) = e^{-\alpha t}\left[B_1 \cos(w_d t) + B_2 \sin(w_d t)\right]$

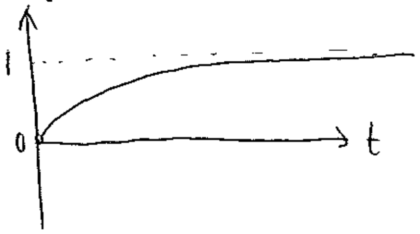
* Overdumped Case

$V_c(0) = 0$

$I_L(0) = 0$ $\Rightarrow$ $V_c(t) = 1 + \frac{s_2}{s_1 - s_2} e^{s_1 t} - \frac{s_1}{s_1 - s_2} e^{s_2 t}$

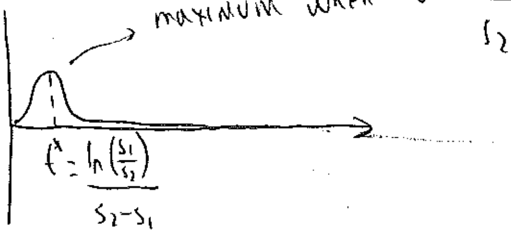
$I_L(t) = \frac{C s_1 s_2}{s_1 - s_2} e^{s_1 t} - \frac{C s_1 s_2}{s_1 - s_2} e^{s_2 t}$

$V_c(t)$



$I_L(t)$

maximum when $t^* = \frac{\ln\left(\frac{s_1}{s_2}\right)}{s_2 - s_1}$



* Critically damped Case

** The system will nearly
** start damped oscillations

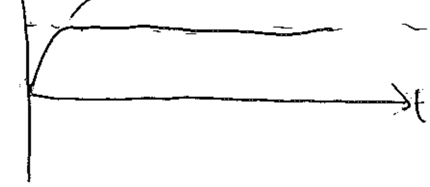
$v_c(0) = 0$

$I_L(0) = 0$

$V_c(t) = 1 + (-1 - \alpha t) e^{-\alpha t}$

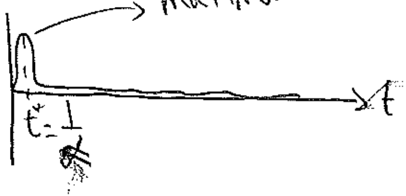
$I_L(t) = \alpha^2 t e^{-\alpha t}$

$V_c(t)$



$I_L(t)$

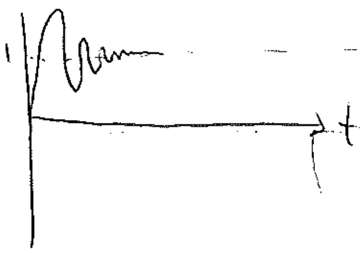
maximum when $= \frac{1}{\alpha}$



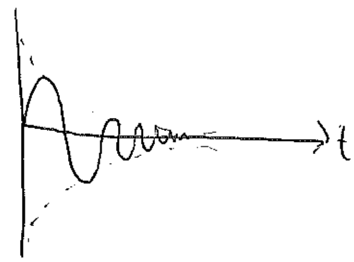
(3)

* Underdamped case $V_c(0)=0$, $I_L(0)=0$

$V_c(t)$



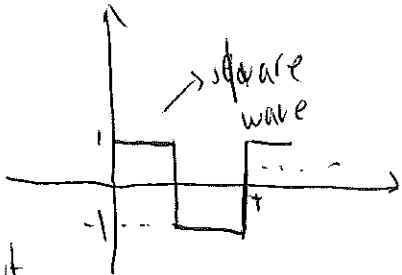
$I_L(t)$



$$V_c(t) = 1 - e^{-\alpha t}\left[\cos(w_d t) + \frac{\alpha}{w_d}\sin(w_d t)\right]$$

$$I_L(t) = C\frac{w_0^2}{w_d}\sin(w_d t)$$

(2) Assume $V_{in}(t)$



f = frequency of input

$\frac{1}{f} = T$ = period of input

If these assumptions are true the square wave can behave similar to a DC source

when $Tk < t < \left(\frac{1+2k}{2}\right)T$ $\quad V_{in}(t) = 1$

$k = 0, \ldots$ $\quad V_{c_{ss}} = 1$

$I_{L_{ss}} = 0$

when $\left(\frac{2k-1}{2}\right)T < t < kT$ $\quad V_{in}(t) = -1$

$k = 1, \ldots$ $\quad V_{c_{ss}} = -1$

$I_{L_{ss}} = 0$

Other assumptions (for steady-state conditions)

* $T \gg \tau_1, \tau_2$

$\tau_{1,2} \rightarrow$ time constants of the RLC circuit

* For overdamped $s = s_{1,2}$ (real negative)

$\tau_1 = \frac{1}{|s_1|}$, $\tau_2 = \frac{1}{|s_2|}$ $\quad s_{1,2} = -\alpha \mp \sqrt{\alpha^2 - w_0^2}$

* For critically damped $s_{1,2} = -\alpha$

$\tau_1 = \frac{1}{|\alpha|}$ $\quad \tau_2 = \frac{1}{|\alpha|}$

* For underdamped $s_{1,2} = -\alpha \mp j\sqrt{w_0^2 - \alpha^2}$

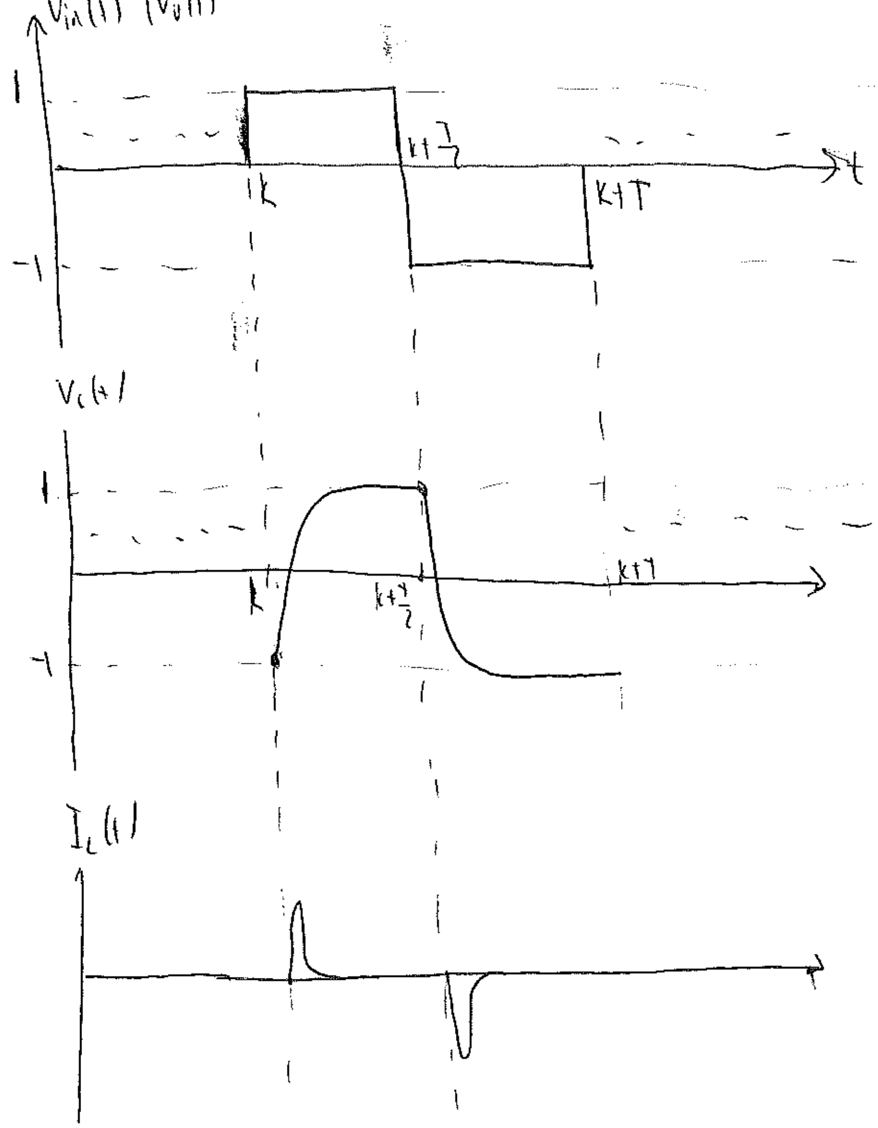
$Re\{s_{1,2}\} = -\alpha \rightarrow$ determines exponential decay

hence $\tau_1 = \frac{1}{|\alpha|}$ $\quad \tau_2 = \frac{1}{|\alpha|}$

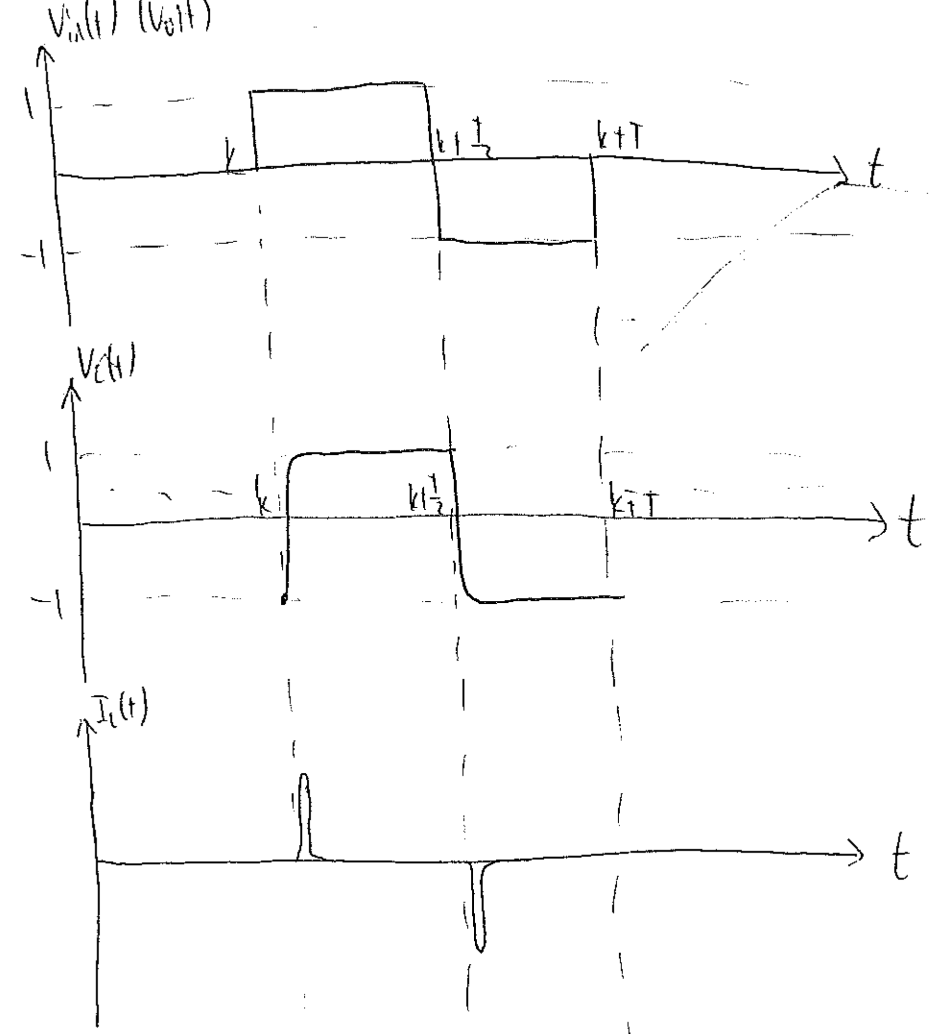
(4)

(3) ⟶ At steady-state conditions with square wave input (when assumptions are satisfied)

* Overdamped



* Critically damped



* Underdamped